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Surprising Geometrical Properties that are Obtained by Transforming any Quadrilateral into a Lattice

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Abstract

The article presents an interesting study of properties existing at any quadrilateral when it develops as a lattice consisting of sub-quadrilaterals with common properties along its rows and columns. Among the properties: quadrilaterals' areas representing arithmetic progression, parallel sections with equal lengths. The study was accompanied by D.G.S. computerized technology. For every property, a mathematical proof of the theorems was given at a level understandable by high school students.

1. Introduction

Research conducted in recent years concerning special properties that exist in different geometrical shapes using mathematical tools [2], [7]. Drawing tools and computerized technology, have produced surprising results that illuminate the beauty of mathematics, and in particular – of Euclidean geometry, and gave stimulus and motivation for extending the research, especially among educators in mathematics and their pre-service teachers [3], [4], [6], [8].

The present paper deals with the investigation of special and surprising properties that are revealed in any quadrilateral what it is turned into a lattice of M × N quadrilaterals. Particular cases have been presented and a generalization was made for the general case.

The investigative activity took place together with the pre-service teachers as a part of a course that dealt with the integration of computerized technology in the teaching of mathematics. GeoGebra applets were prepared for investigating the properties, and some of the proofs presented where prepared by the students. One can find more on the importance of using computerized technology in [1], [5], [8], [9].

2. From a quadrilateral to a lattice

From the quadrilateral ABCD which is the basic cell, one goes over to a quadrilateral with N rows and M columns, as shown in Figure 1.
The new structure, whose shape is a quadrilateral, and is called a lattice, is obtained from the original quadrilateral, as described below.

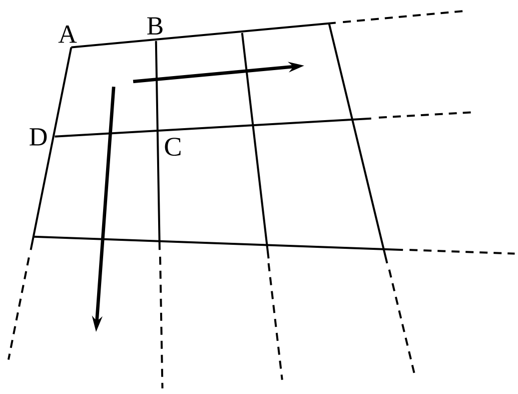


Figure 1

Expansion from a quadrilateral to a lattice

Lemma 1

When straight lines connect the middles of the opposite sides in any quadrilateral ABCD, the sum of the areas of two opposite quadrilaterals is equal to the sum of the areas of the other pair of opposite quadrilaterals (Figure 2).

Prove that: $S_1 + S_3 = S_2 + S_4$.

The proof of the lemma is without words and is based on the property that the median in a triangle divides its area into two triangles with equal areas:

$$S_1 + S_3 = S_2 + S_4 = \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d}$$

In order to illustrate the property according to which the sum of the opposite quadrilaterals always equals half the area of the quadrilateral ABCD, we construct a GeoGebra applet in which one can drag each of the vertices of the quadrilateral ABCD. For each location of the vertices (including a concave quadrilateral), the screen shows the sum of the areas of the opposite quadrilaterals.

Link to applet 1: http://tube.geogebra.org/material/simple/id/3238811

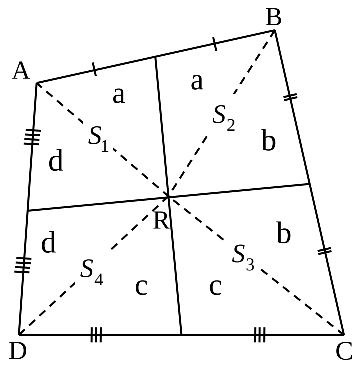


Figure 2

The sum of the areas of opposite quadrilaterals

Theorem 1

In some quadrilateral ABCD, a pair of opposite sides was divided into three equal parts, and connected by straight lines, as shown in figure 3 (AM = MN = NB, DP = PQ = QC). Three quadrilaterals were obtained; whose areas are:

$$S_1 = S_{\square \mathrm{AMPD}}$$

$$S_2 = S_{\square \mathrm{MNQP}}$$

$$S_3 = S_{\square \mathrm{NBCQ}}$$

Prove that these areas form an arithmetic progression, in other words: $S_1 + S_3 = 2 \cdot S_2$.

Proof

From the points A, M and N, we drop perpendiculars to the straight line DC. From this we obtain that the quadrilateral ANN_1A_1 is a right-angled trapezoid whose bases are $h_1 = AA_1$ and $h_3 = NN_1$, and where
$h_2 = MM_1$ is a midline, and therefore $h_1 + h_3 = 2h_2$. Since the bases of the triangles have equal lengths: DP = PQ = QC, the areas of the hatched triangles in Figure 3 satisfy:

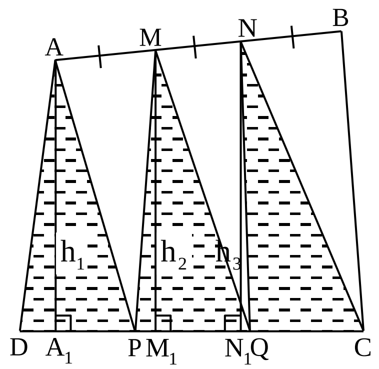


Figure 3

Calculation of the areas of the quadrilaterals

$2S_{\Delta\,\mathrm{MPQ}} = S_{\Delta\,\mathrm{ADP}} + S_{\Delta\,\mathrm{NCQ}}$, in other words – the areas of the triangles form an arithmetic progression. In the same manner we prove that $2S_{\Delta\,\mathrm{MNQ}} = S_{\Delta\,\mathrm{AMP}} + S_{\Delta\,\mathrm{NBC}}$. From both relations it follows that: $S_1 + S_3 = 2\cdot S_2$.

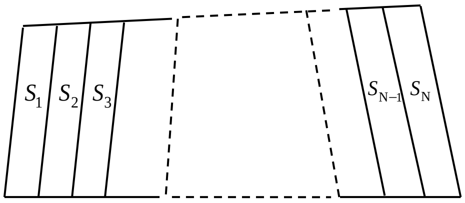


Figure 4

Calculation of the areas

Conclusion from Theorem 1

If one divides to opposite sides of a quadrilateral, each into N segments of the quote lengths, one obtains a lattice of the order “N×1”, as described in figure 4.

The areas of this quadrilateral, S_1 , S_2 , ... , S_N, form an arithmetic progression.

The proof is based on Theorem 1.

When N is odd, the area of the middle quadrilateral relative to area of the original quadrilateral is $\frac{1}{\mathrm{N}}$.

When N is even, there are two quadrilaterals at the center of the original quadrilateral. The sum of their areas with respect to the area of the original quadrilateral is $\frac{2}{\mathrm{N}}$.

To illustrate this property, we prepared an applet that includes two toolbars, one for an odd N, and one for an even N, in which one can change the value of N using a toolbar, and obtain the relative area of the middle quadrilateral (or the two middle quadrilaterals). Of course, at first sight the result is surprising, but after some thought one realizes that this is the property of an arithmetic progression.

Link to applet 2: https://www.geogebra.org/material/simple/id/3238991

Note: Applet 2 shall also be used to illustrate the property of Theorem 5 below.

Theorem 2

Given is a quadrilateral ABCD, in which:

$S_3 \geq S_1$, $S_4 \geq S_2$, as shown in Fig. 5. We extend the length of the sides of the quadrilateral by a factor of two outwards, and obtain the points B_1, C_1, C_2, D_2. Then there holds:

$$S_{\mathrm{BB_1C_1C}} - S_{\mathrm{ABCD}} = 2\cdot(S_3 - S_1)$$

$$S_{\mathrm{DCC_2D_2}} - S_{\mathrm{ABCD}} = 2\cdot(S_4 - S_2)$$

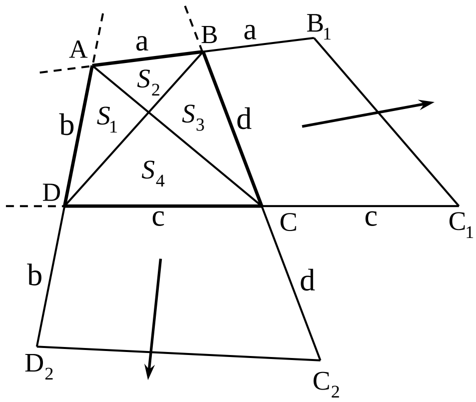
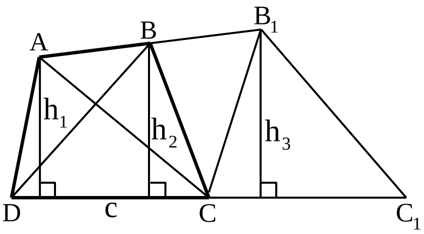


Figure 5

Calculation of the areas

Proof

We copy the quadrilateral AB_1C_1D, and draw in it the altitudes h_1, h_2, h_3, as shown in figure 6.



$DC = CC_1 = c$

$$S_{BCC_1B_1} - S_{ABCD} = \underbrace{(S_{\Delta BCB_1} - S_{\Delta ABC})}_{=0 \text{ because } AB=BB_1} + (S_{\Delta CB_1C_1} - S_{\Delta ADC}) =$$

$$= S_{\Delta CB_1C_1} - S_{\Delta ADC} = \frac{c \cdot (h_3 - h_1)}{2}$$

Since $2h_2 = h_1 + h_3$ (midline in a trapezoid), we have $h_3 = 2h_2 - h_1$, and therefore:

$$S_{BCC_1B_1} - S_{ABCD} = c \cdot (h_2 - h_1) = [(S_3 + S_4) - (S_1 + S_4)] \cdot 2 = 2 \cdot (S_3 - S_1)$$

In the same manner we obtain: $S_{DCC_2D_2} - S_{ABCD} = 2 \cdot (S_4 - S_2)$.

Conclusion 2

The initial cell ABCD, as the first element in a progression, determines the common difference of the arithmetic progression in the horizontal direction (to the right) or in the vertical direction (downwards). In the expansion "to the right", the common difference of the progression is $2 \cdot (S_3 - S_1)$, and in the expansion downwards, the common difference is $2 \cdot (S_4 - S_2)$.

Theorem 3

In the quadrilateral ABCD, is given that:

$$\frac{AM}{MB} = \frac{DP}{PC} = \alpha, \ \frac{AQ}{QD} = \frac{BN}{NC} = \beta$$

The segments MP and QN intersect at the point R, as shown in figure 7.

Then there holds: $\frac{MR}{RP} = \beta$, $\frac{QR}{RN} = \alpha$

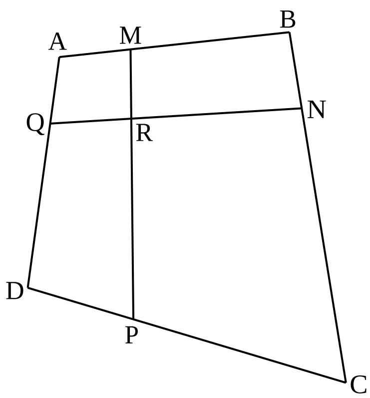


Figure 7

Calculation of coordinates in a lattice

Proof

We denote the *x*-coordinates of the vertices of the quadrilateral ABCD by x_A, x_B, x_C, x_D, and through them we express the *x*-coordinates of the points M, P, Q, N.

$x_M = \frac{x_A + \alpha x_B}{\alpha + 1}$, $x_P = \frac{x_D + \alpha x_C}{\alpha + 1}$, $x_Q = \frac{x_A + \beta x_D}{\beta + 1}$, $x_N = \frac{x_B + \beta x_C}{\beta + 1}$ Now we calculate the *x*-coordinate of the point R_1 that divides the segment MP by a ratio of $\frac{MR_1}{R_1P} = \beta$, and obtain: $x_{R_1} = \frac{x_M \cdot 1 + x_P \cdot \beta}{\beta + 1} = \frac{x_A + \alpha x_B + \beta x_D + \alpha\beta x_C}{(\alpha+1)(\beta+1)}$.

In a similar manner we calculate the coordinate x_{R_2} of the point R_2 that divides the segment QN by the ratio $\frac{QR_2}{R_2N} = \alpha$. We obtain $x_{R_2} = x_{R_1}$.

In other words, R_1 and R_2 coincide at one-point R. In the same manner would prove that there also holds $y_{R_2} = y_{R_1}$.

Conclusion 3

When, in the quadrilateral ABCD, the opposite sides AB and CD are divided into M equal parts, and the opposite sides AD and BC are divided into N equal parts, a lattice is obtained as described in Figure 8. Each horizontal "bar" in the lattice is divided into M equal parts, and each vertical "bar" in the lattice is divided into N equal parts.

It could be said that the "cell" at the upper left corner (by the vertex A) is "inflated" into a quadrilateral ABCD, which is an N×M lattice.

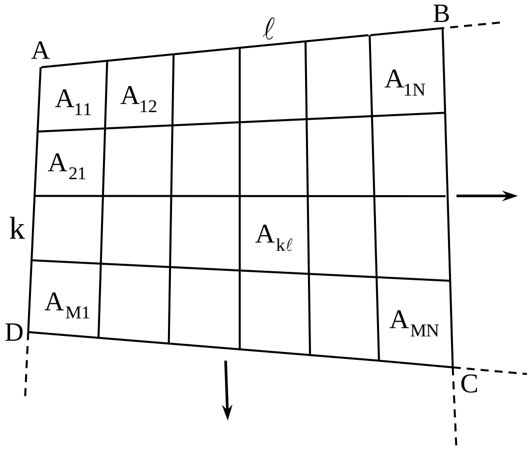


Figure 8

The areas comprising the lattice

Theorem 4

The cells in the lattice shown in Figure 8 were numbered in accordance with row k and column *l*, and the area of the corresponding cell was marked by A_{kl}. then:

a) The areas of the cells **in each row** form an arithmetic progression, where the **common** difference in each row **is equal**.

b) The areas of the cells **in each column** form an arithmetic progression, where the **common** difference in each column **is equal**.

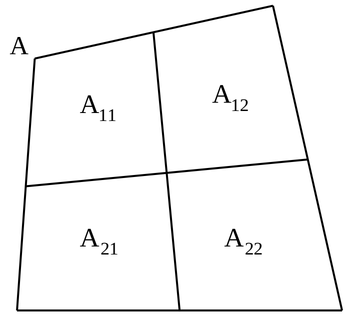


Figure 9

The areas of a sub-quadrilateral

Proof of a and b

Figure 9 shows by the vertex A.

From Lemma 1, we have: $A_{12} + A_{21} = A_{11} + A_{22}$.

From this it follows that $A_{12} - A_{11} = A_{22} - A_{21}$, therefore a) is proven.

From this it also follows that $A_{21} - A_{11} = A_{22} - A_{12}$, therefore b) is proven.

From Theorem 2, the common difference D_1 in each row is:

$D_1 = 2 \cdot (S_3 - S_1)$,

and the common difference in each column is: $D_2 = 2 \cdot (S_4 - S_2)$, as shown in Figure 10, which shows the enlarged cell A_{11}.

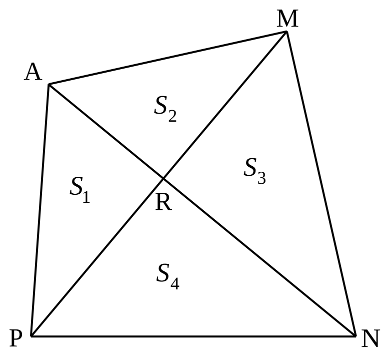


Figure 10

The areas of a sub-quadrilateral

Lemma 2

In quadrilateral A_{11} the diagonals meet at the point R.

When: $D_1 = D_2$ then R is the middle of the diagonal PM (Figure 10).

Proof

$D_1 = D_2 \;\Rightarrow\; S_3 - S_1 = S_4 - S_2 \;\Rightarrow\; S_3 + S_2 = S_1 + S_4$

Hence it follows that $S_{\Delta APN} = S_{\Delta AMN}$. Since both triangles have a common side, their altitudes from the vertices M and P to the base AN are equal, and therefore MR = RP.

Conclusion 4
If $D_1 = D_2$ for each $1 \le k \le N$ and $1 \le \ell \le M$, in each cell A_{kl} one of the diagonals is bisected by the other one. In other words, the property PR = RM spreads to all the cells, since the cell A_{kl} is built from the cell A_{11}.

Property in a 2×2 lattice
The quadrilateral ABCD was divided into a 2×2 lattice, where the quadrilateral ("cell") A_{11} (Figure 11) satisfies the condition that the diagonal AR bisects the diagonal QM (in other words, $D_1 = D_2$, as shown in Figure 11).

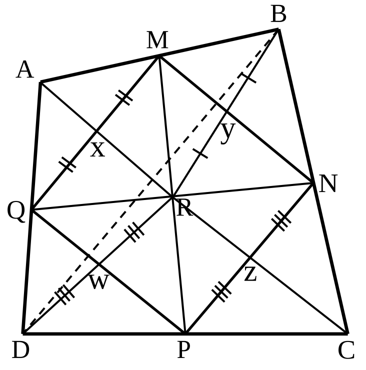


Figure 11
Property in a 2×2 lattice

Lemma 3
Under these conditions the diagonal AC passes through the point R and bisects the diagonal BD.
Proof
In Conclusion 4 it was proven that if the property $D_1 = D_2$ holds in cell A_{11}, then it holds in each cell A_{kl}.
Therefore, in this case, the points x, y, z, w are the middles of the segments QM, BR, NP and RD, respectively, as shown in the Figure 12. The quadrilateral QMNP is a parallelogram (connecting the middles of the sides of a quadrilateral), and RC and RA are also parallel to MN, therefore the points A, R, C are located on a straight line. Hence it also follows that BD is bisected by AC (the segment AR is a part of AC, bisects QM and therefore also BD).

Conclusion 5
The property that in A_{11} one of the diagonals is bisected by the other is inherited by each partial lattice of the order 2×2. Hence one can generalize that for any lattice of the order N×N (square lattice), if the cell A_{11} has the property that $D_1 = D_2$, and one of the diagonals bisects the other, then the property is inherited by each square sub-lattice.

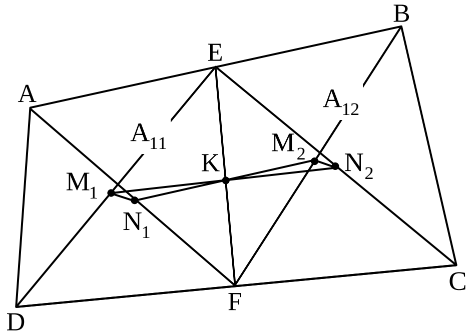


Figure 12
Parallel segments of equal length

Theorem 5
ABCD is a lattice of the order 1×2 comprised of the cells A_{11} and A_{12}, as shown in the figure 12. Let M_1 and N_1 be the mid-points of the diagonals ED and AF in the cell A_{11} and let M_2 and N_2 be the mid-points of the diagonals BF and EC of the cell A_{12}.
Then there holds:

a) $M_1N_1 = M_2N_2$
b) $M_1N_1 \parallel M_2N_2$

Note: If ABCD is a trapezoid, then all four points M_1, N_1, M_2, N_2 are on the same straight line.
Proof
N_1M_2 is a midline in the triangle ΔAFB, therefore:

1) $N_1M_2 \parallel AB$
2) N_1M_2 bisects EF at the point K (property of the midline), and is bisected by EF, and therefore: $N_1K = KM_2$.

In the same manner, the segment M_1N_2 is a midline in the triangle ΔDEC, and it also bisects and is bisected by EF at the same point K, therefore $N_2K = M_1K$.

Hence it follows that the quadrilateral $M_1N_1N_2M_2$ is a parallelogram, and therefore: $M_1N_1 = M_2N_2$, and also $M_1N_1 \parallel M_2N_2$.

In applet 2 one can see that the segments that connect the middle points of the diagonals of each sub-quadrilateral have equal lengths and are parallel to each other.

Conclusion 6

In each sub-lattice the order 1×1 of a lattice of the order M×N, **the distance** between the middles of its diagonals conserved. In addition, all the segments that connect the diagonals of a sub-lattice of the order 1×1 are parallel.

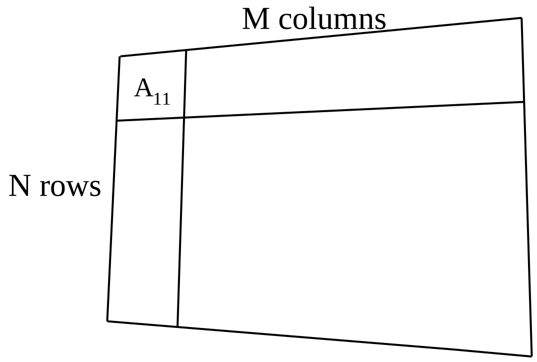


Figure 13

The mid-points of the diagonals of a lattice

Theorem 6

In a lattice of the order N×M (Figure 13), let k be the distance between the middles of the diagonals of the quadrilateral A_{11}. Then the distance K to the middles of the diagonals of the lattice N×M is M·N·k.

Proof

For the purpose of the proof we focus on a lattice with a single row $A_{11}, A_{12}, \dots, A_{1M}$, where the x coordinates of the vertices of the quadrilateral A_{11} are marked as described Figure 14.

We calculate the values the coordinates of the vertices x_R and x_{M+1} using the values of the vertices x_1, x_2, x_3 and x_4. As proven, $x_1, x_2, \dots x_{M+1}$ form an arithmetic progression whose common difference is $x_1 - x_2$.

Therefore: $x_{M+1} = Mx_2 - (M-1)x_1$, and similarly $x_R = Mx_3 - (M-1)x_4$.

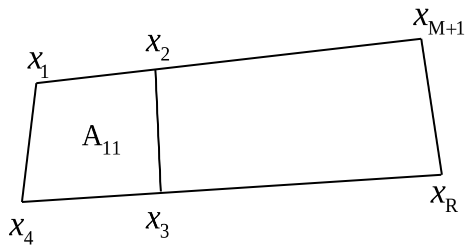


Figure 14

The coordinates of the middle of a segment in a lattice sub-quadrilateral

The coordinates of the center of the diagonal that connects the vertex x_4 with the vertex x_{M+1} is: $x_1 = \frac{x_4 + Mx_2 - (M-1)x_1}{2}$. In the same manner, the x coordinate of the middle of the other diagonal is $x_2 = \frac{x_1 + Mx_3 - (M-1)x_1}{2}$.

From this we obtain: $(x_1 - x_2)^2 = \left(\frac{Mx_4 + Mx_2 - Mx_3 - Mx_1}{2}\right)^2 = M^2\left(\frac{x_4 + x_2}{2} - \frac{x_1 + x_3}{2}\right)^2$,

and similarly for the coordinate y.

Therefore, the distance between the two mid-points is K = M·k.

Therefore, when the first row is considered as a 1×M sub-lattice, then for N rows we obtain K = M·N·k.

Conclusion 7

In each sub-lattice of the order M × N there holds $K_{M\times N} = k\cdot M\cdot N$, where k is the distance between the middles of the diagonals of the basic quadrilateral A_{11}.

An applet was prepared, which presents at two-dimensional lattice in which two toolbars can be used to change the numbers of the rows and the columns. One can drag each of the vertices of the basic quadrilateral, thus changing its sides, since the areas in each of the rows constitute an arithmetic progression, and similarly the areas of the columns. The common differences of the areas of the rows

and the columns are shown on the screen. The screen also shows the connection between the distance between the mid-points of the external quadrilateral and the distance between the mid-points of the corner sub-quadrilateral (left corner top row).
Link to applet 3: https://www.geogebra.org/material/simple/id/3239081

3. Summary

We presented an interesting investigation of surprising geometrical properties that are revealed during the extension of any quadrilateral into an M×N lattice.
At the first stage we presented properties that exist in the original quadrilateral ("the basic cell"), with a subsequent generalization made for properties conserved during the transition to a lattice.
At each stage we presented mathematical proofs that rely on basic knowledge of geometry.
The results serve as a motive to continue investigation and to discover additional properties.

Acknowledgement
The authors thank Mr. Idan Tal for his enthusiasm for the preparation of the GeoGebra applets.

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5. Appendix- Verbal description to the Applets

Applet 1: The areas of two opposite quadrilaterals

The applet illustrates the property according to which the sum of the opposite quadrilaterals always equals half the area of the quadrilateral ABCD. We construct a GeoGebra applet in which one can drag each of the vertices of the quadrilateral ABCD, and for each location of the vertices (including a concave quadrilateral), the screen shows the sum of the areas of the opposite quadrilaterals.

Applet 2: The relative area of the middle quadrilateral

To illustrate the property of relative areas of the middle and center quadrilateral, the applet includes two toolbars, one for an odd N, and one for an even N, in which one can change the value of N using a toolbar, and obtain the relative area of the middle quadrilateral (or the two middle quadrilaterals). The result is surprising, it just a property of an arithmetic progression.

The applet is also to show that the segments that connect the middle points of the diagonals of each sub-quadrilateral have equal lengths and are parallel to each other.

Applet 3: Properties in two-dimensional lattice

The applet presents at two-dimensional lattice in which two toolbars can be used to change the numbers of the rows and the columns. One can drag each of the vertices of the basic quadrilateral, thus changing its sides, since the areas in each of the rows constitute an arithmetic progression, and similarly the areas of the columns. The common differences of the areas of the rows and the columns are shown on the screen. The screen also shows the connection between the distance between the mid-points of the external quadrilateral and the distance between the mid-points of the corner sub-quadrilateral (left corner top row).